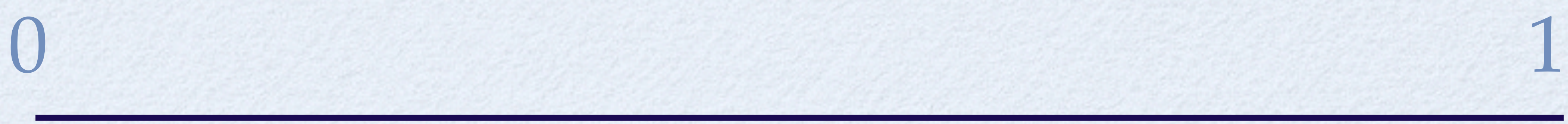
The Cantor Set

The Construction



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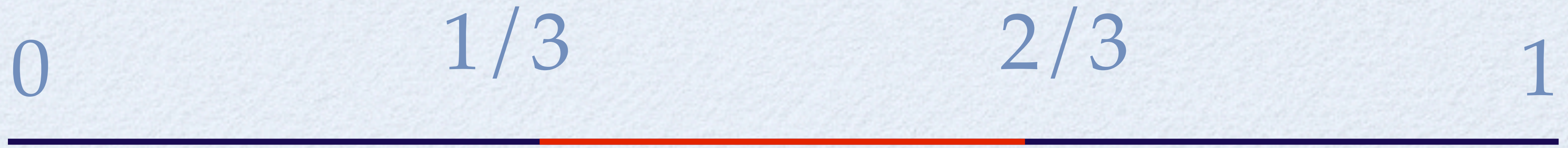

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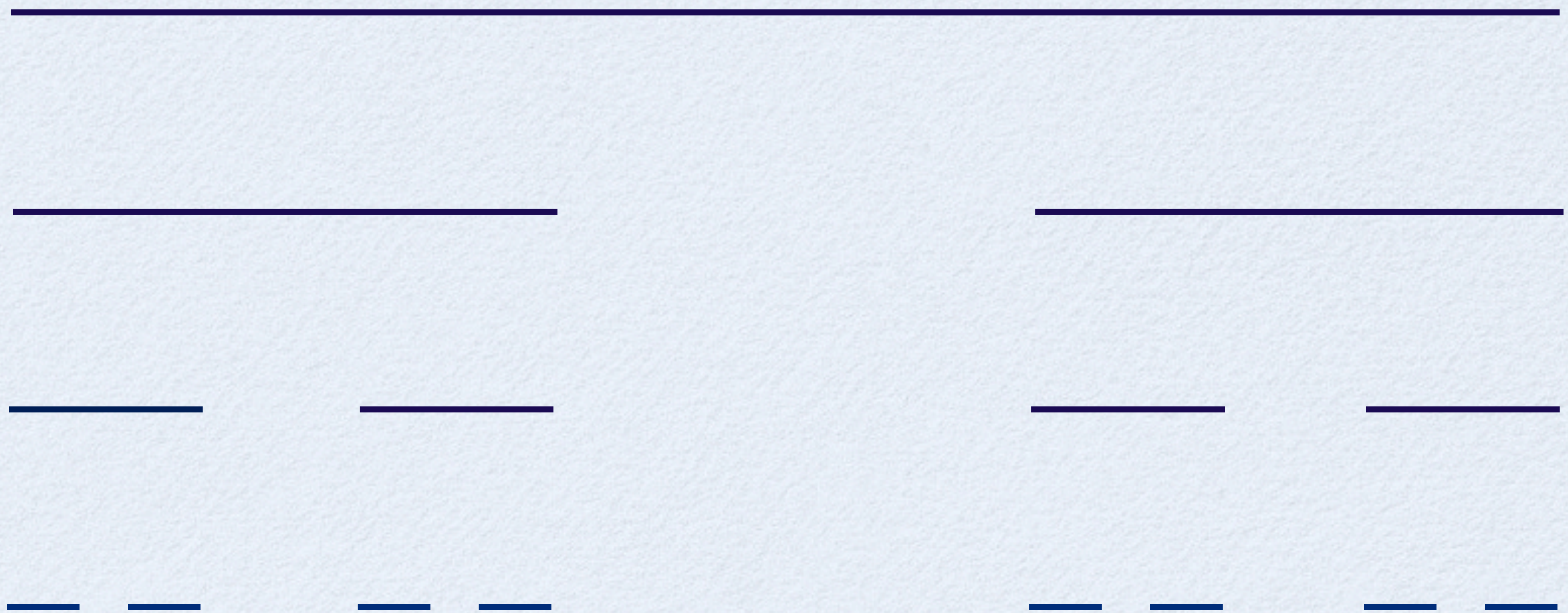

and so on...

The set we're interested in is
the limit of this process.

The set of points that remain after all these "middle thirds" have been deleted is called the Cantor set .

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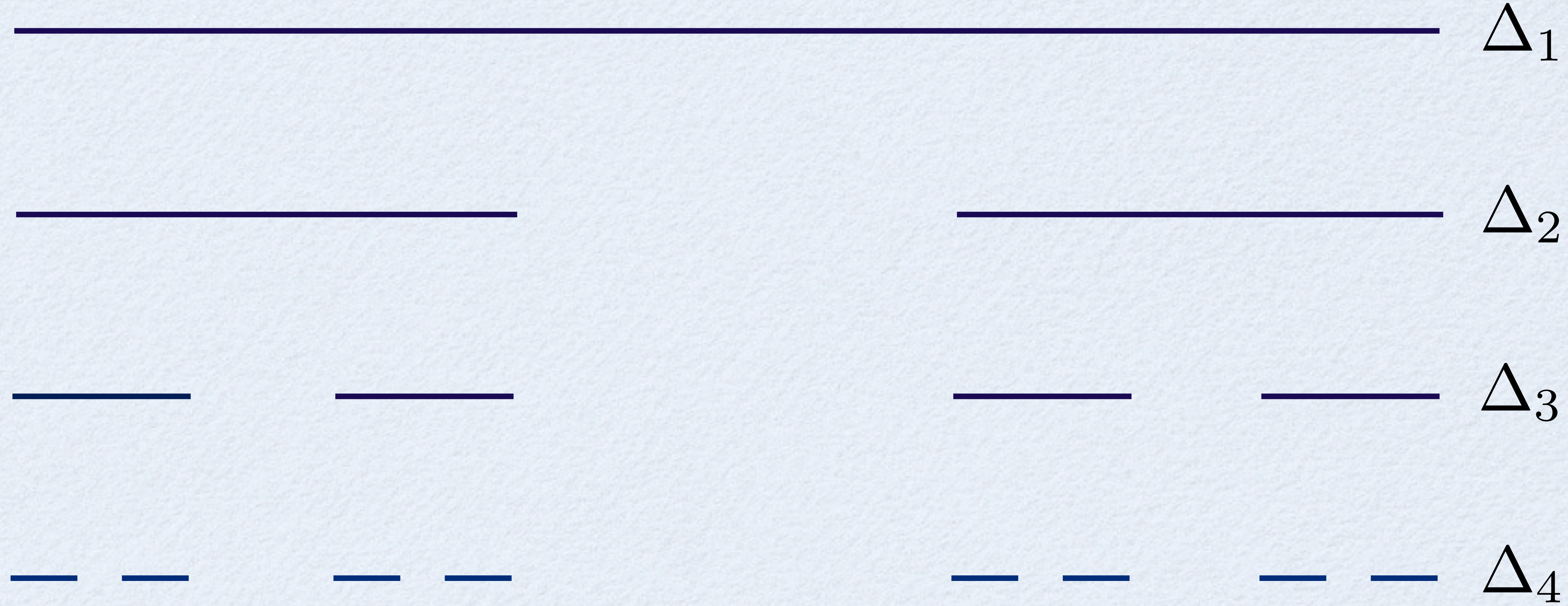
(Georg Cantor, 1845-1918)

It's probably not at all clear that any points remain but, as we'll see, there are tons of points in the Cantor set!

In spite of the fact that it's hard to visualize the Cantor set, it's not hard to understand why it must be a very big set.

The Construction



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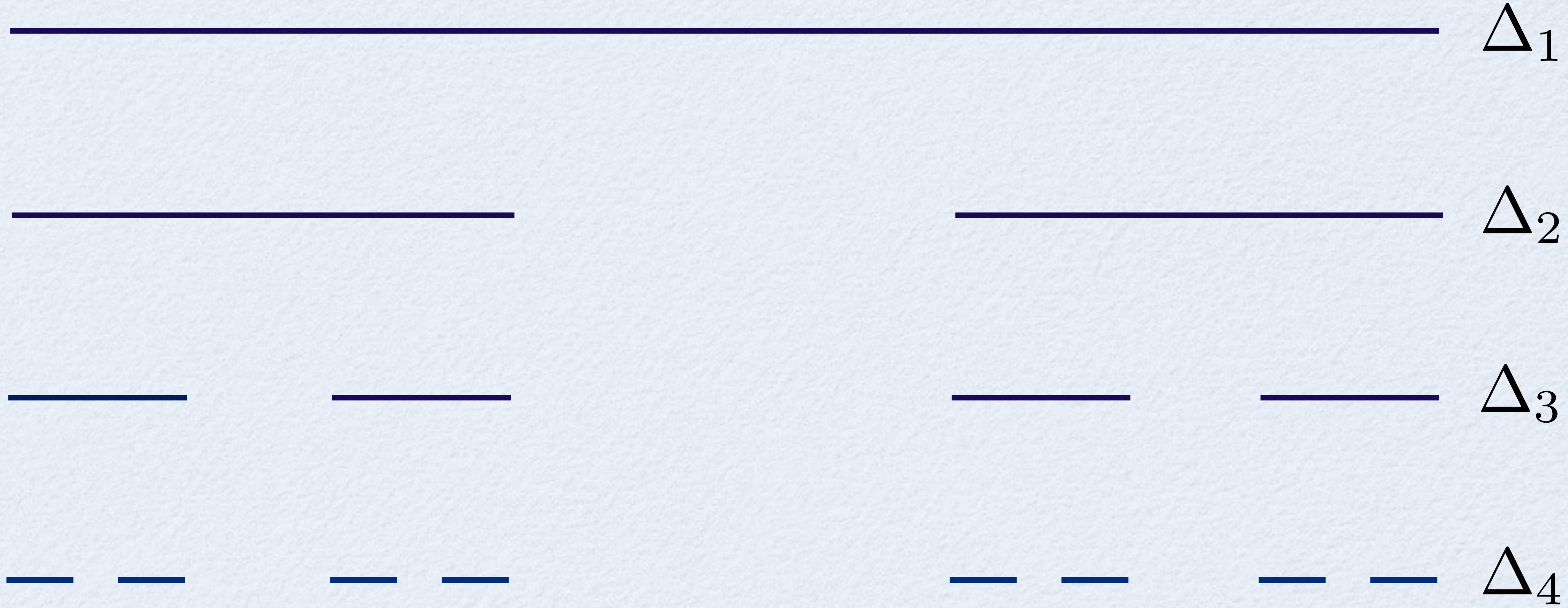
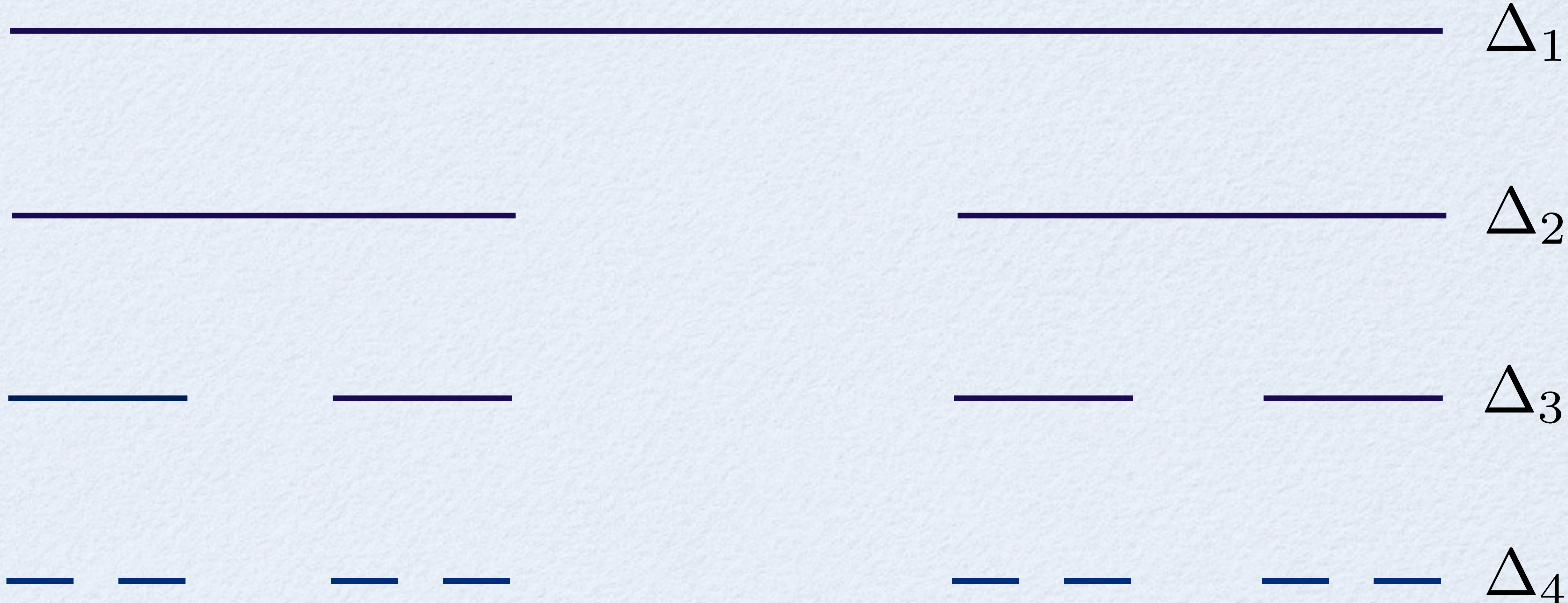


Formally:

$$\Delta = \bigcap_{n=1}^{\infty} \Delta_n$$

THE CONSTRUCTION



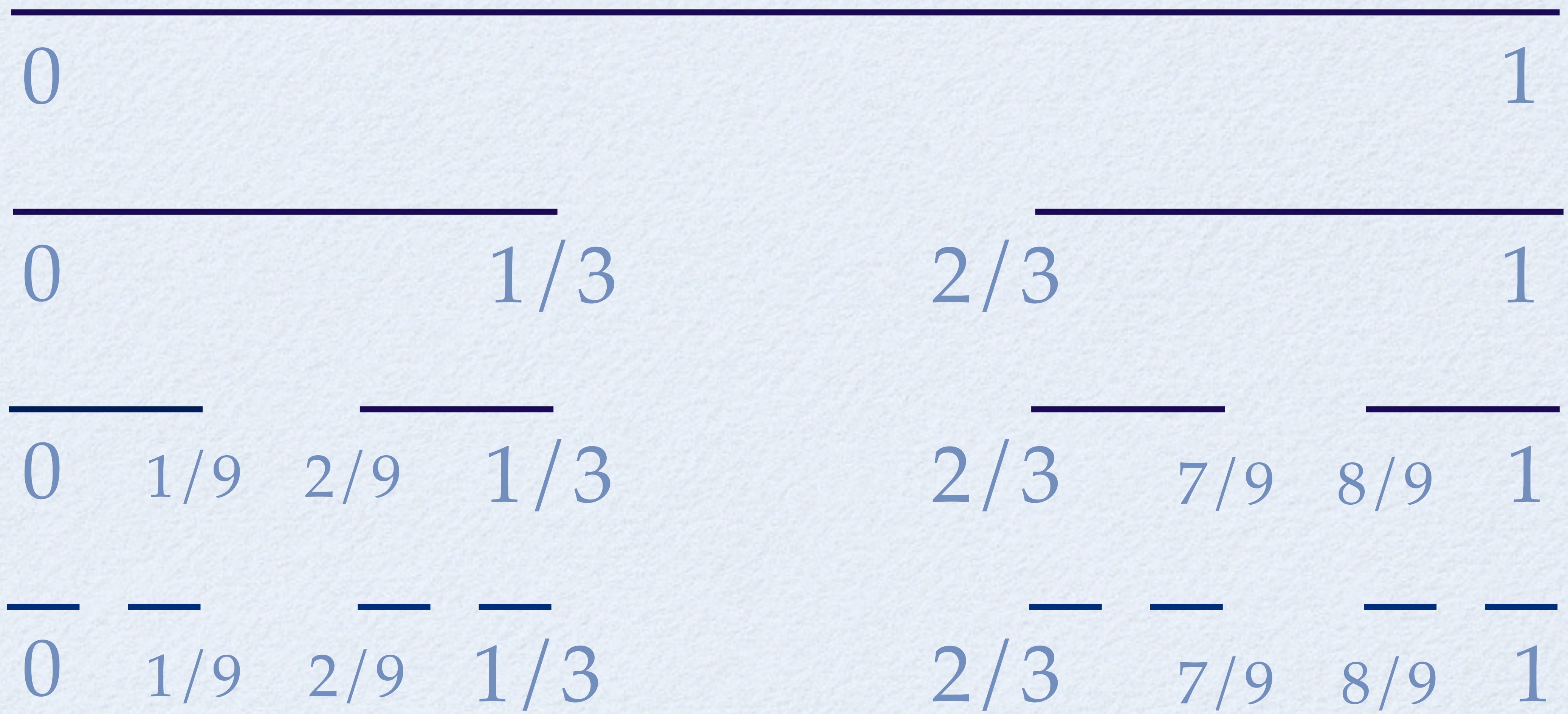
Also:

$$\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots \supset \Delta_n \supset \cdots$$

So Δ is the limit of the Δ_n's.

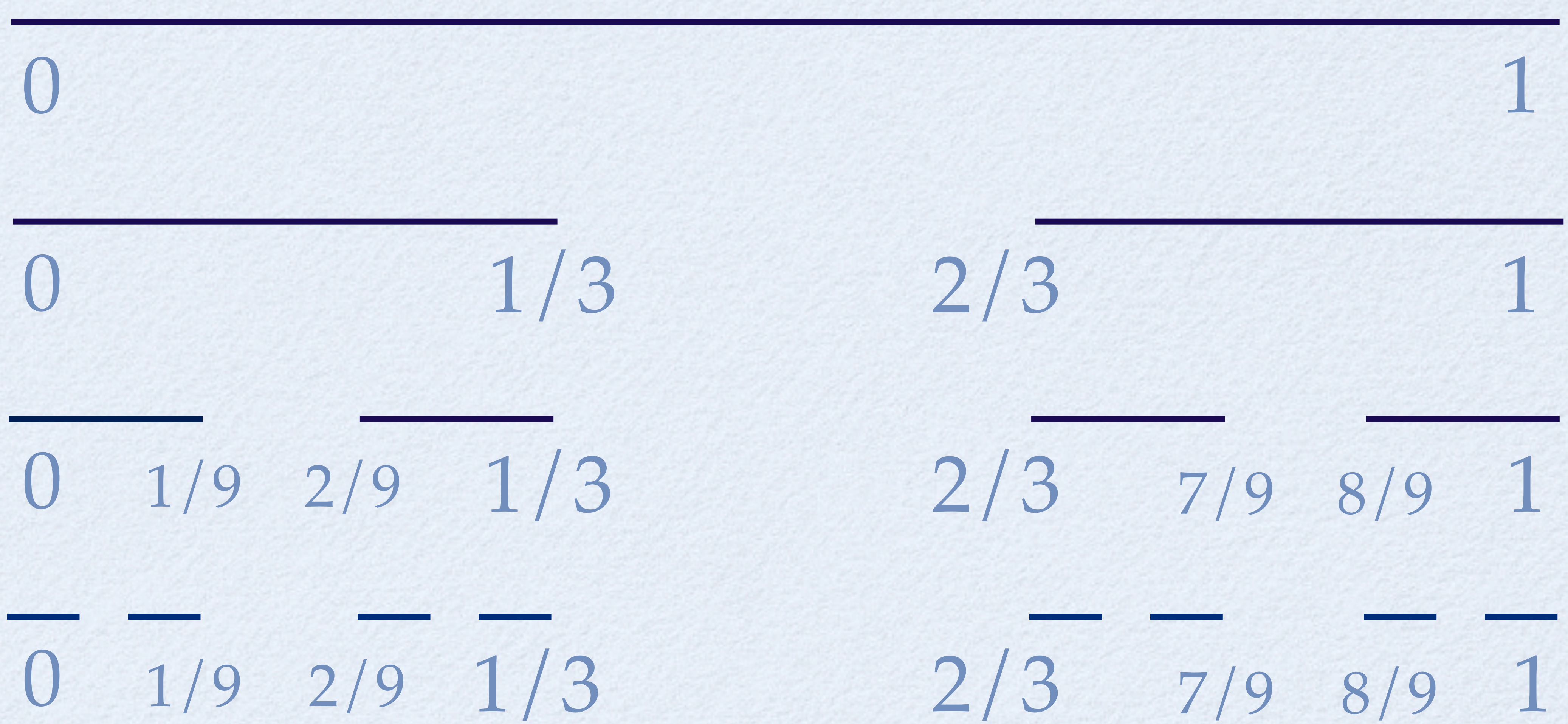
To see that we haven't just given an elaborate definition for an empty set, let's recall again (a portion of) our initial construction, this time labeling the endpoints.

The Construction



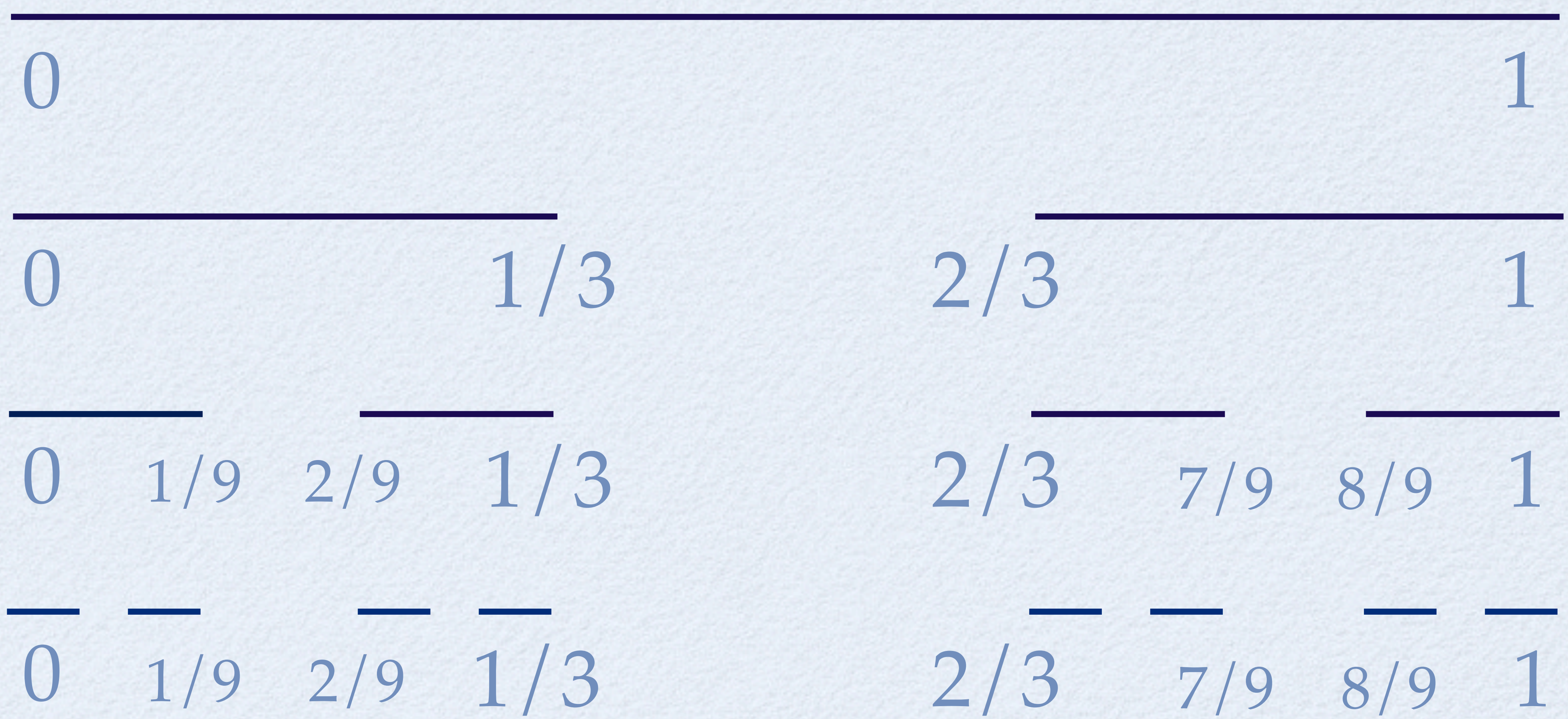
Since we always remove points from the middle of an interval, notice that the endpoints 0 and 1 remain after the first "deletion".

The Construction



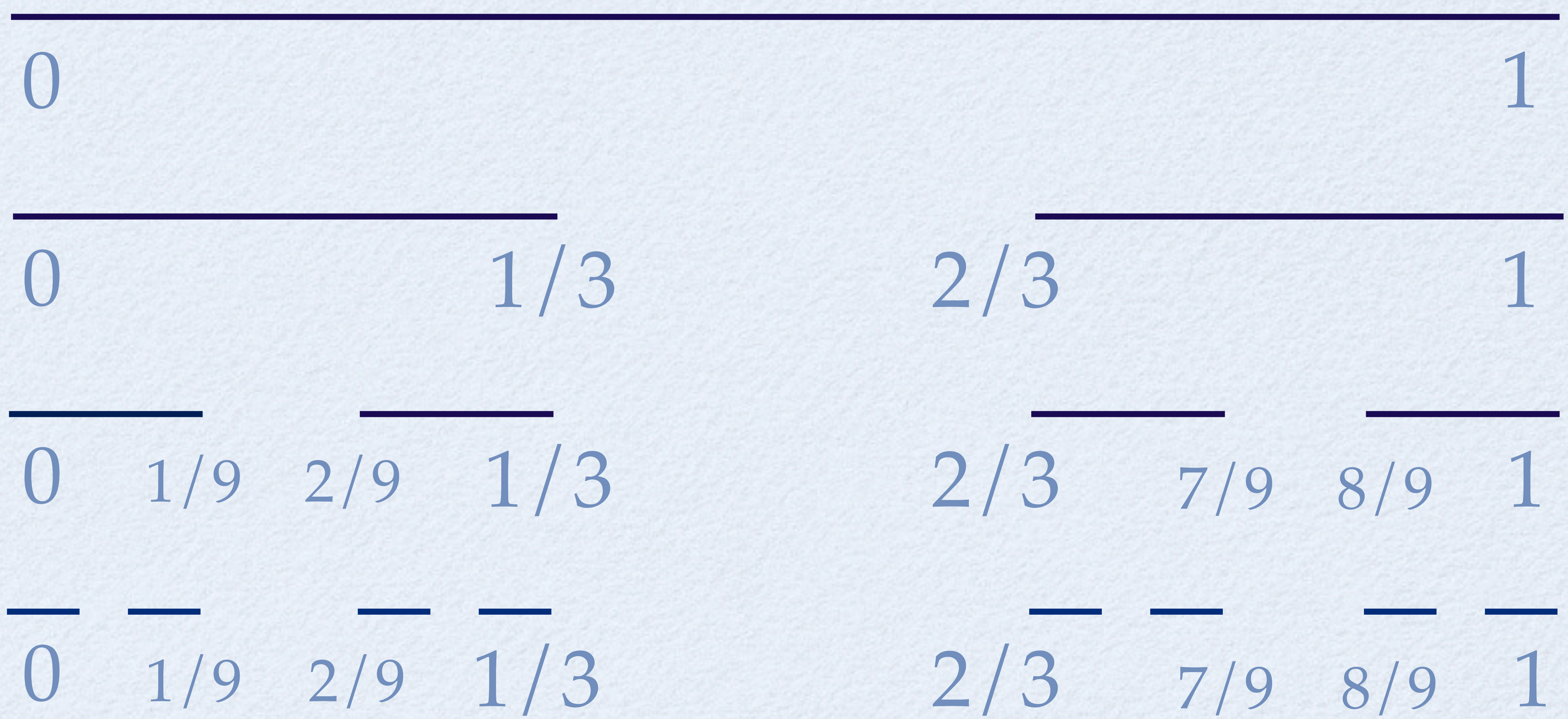
Both 0 and 1 remain after the second and third deletions, too. In fact, they remain after every subsequent deletion.

THE CONSTRUCTION



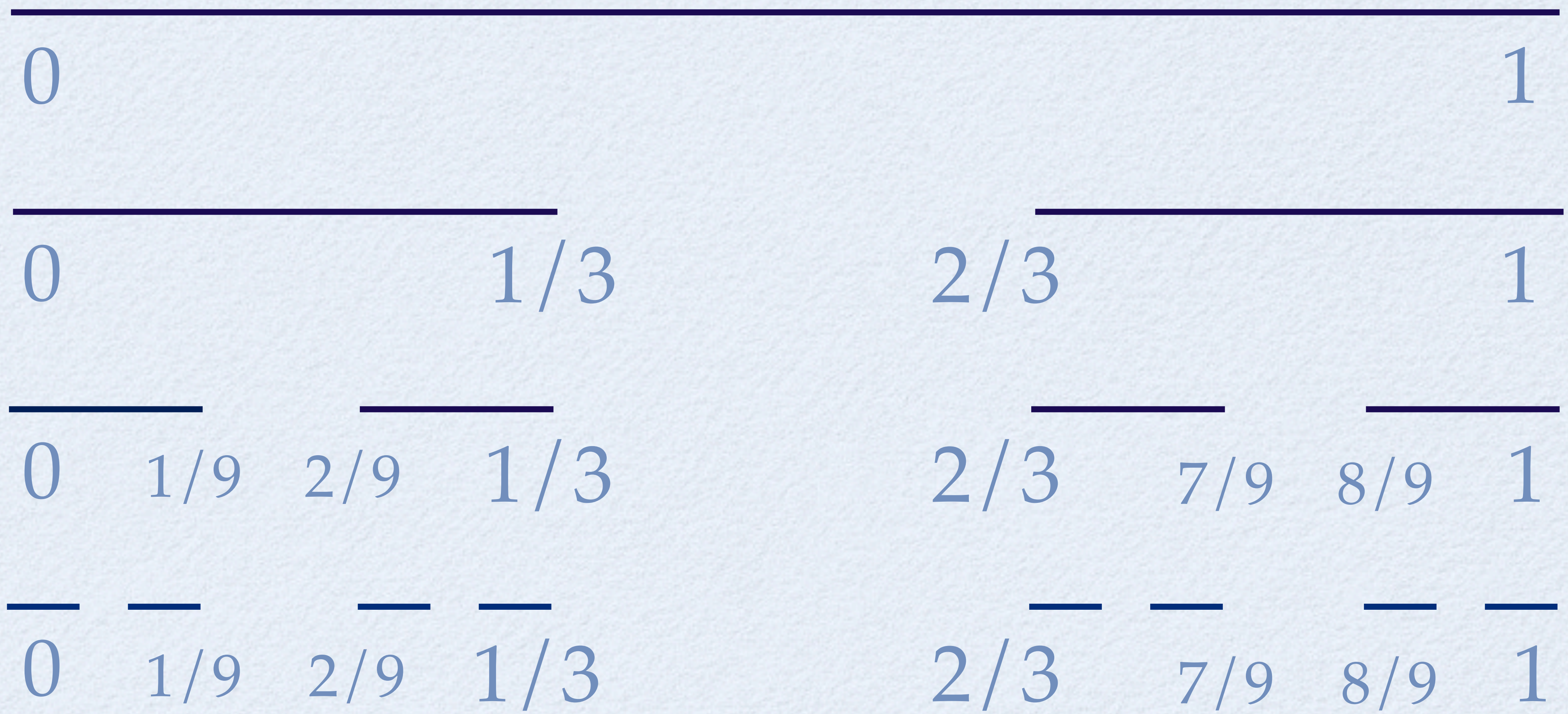
Thus, the limiting set contains at least the two endpoints 0 and 1.

THE CONSTRUCTION



But now we just apply the same reasoning to the points 1/3 and 2/3. Notice that they, too, remain in our set after any deletion.

THE CONSTRUCTION



The same is true of 1/9, 2/9, 7/9, and 8/9...

In short, the Cantor Set will necessarily contain the endpoints of any "discarded middle thirds" intervals; namely,

0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, ...

Thus the Cantor Set consists of

an infinite number of points!

In fact: The Cantor Set consists of

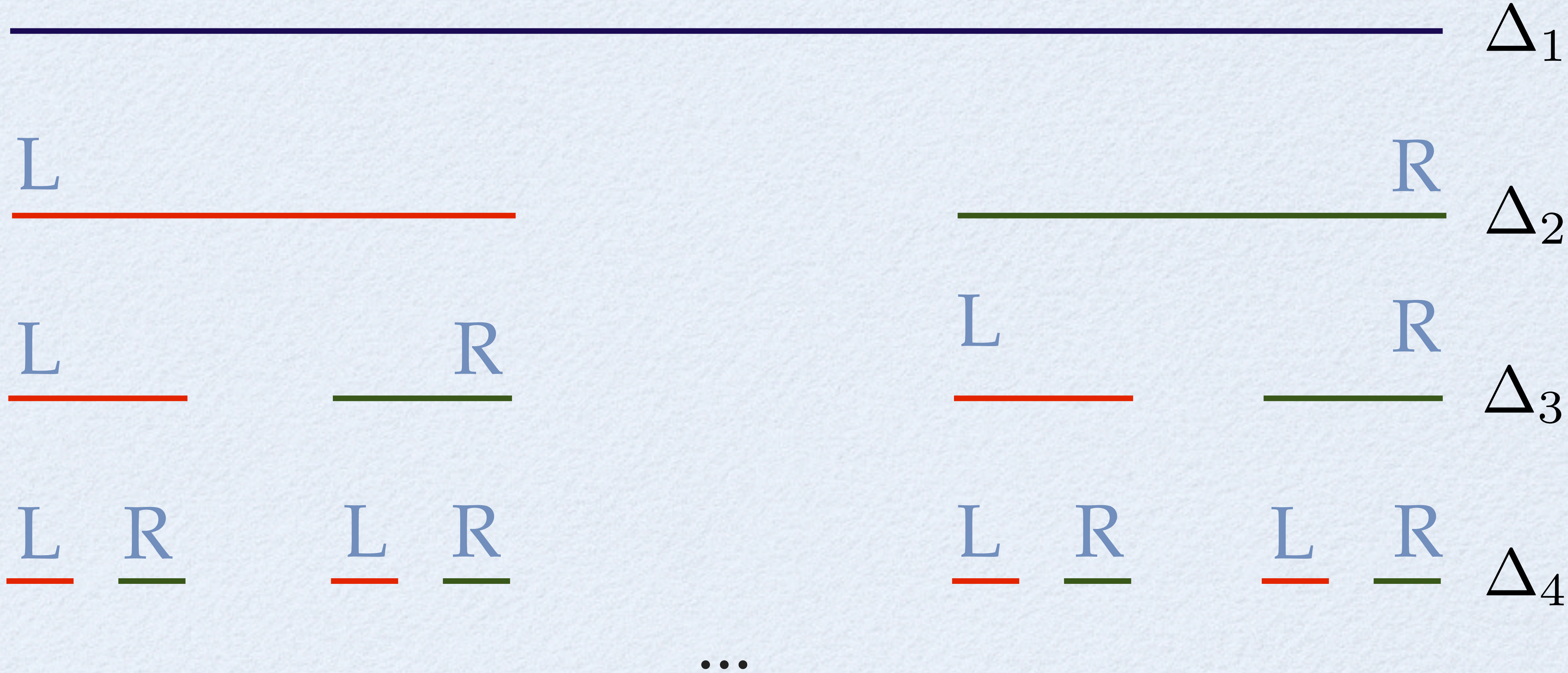
an uncountably infinite

number of points!

To prove this, we set up a correspondence, or "matching", between the points in the Cantor set and the points in the interval [0,1].

In order to do this, we'll take a fresh look at our construction.

The Construction



Instead of removing middle thirds, imagine our construction as a process of retaining the left and right thirds of each interval, labeling our choices as we go along.

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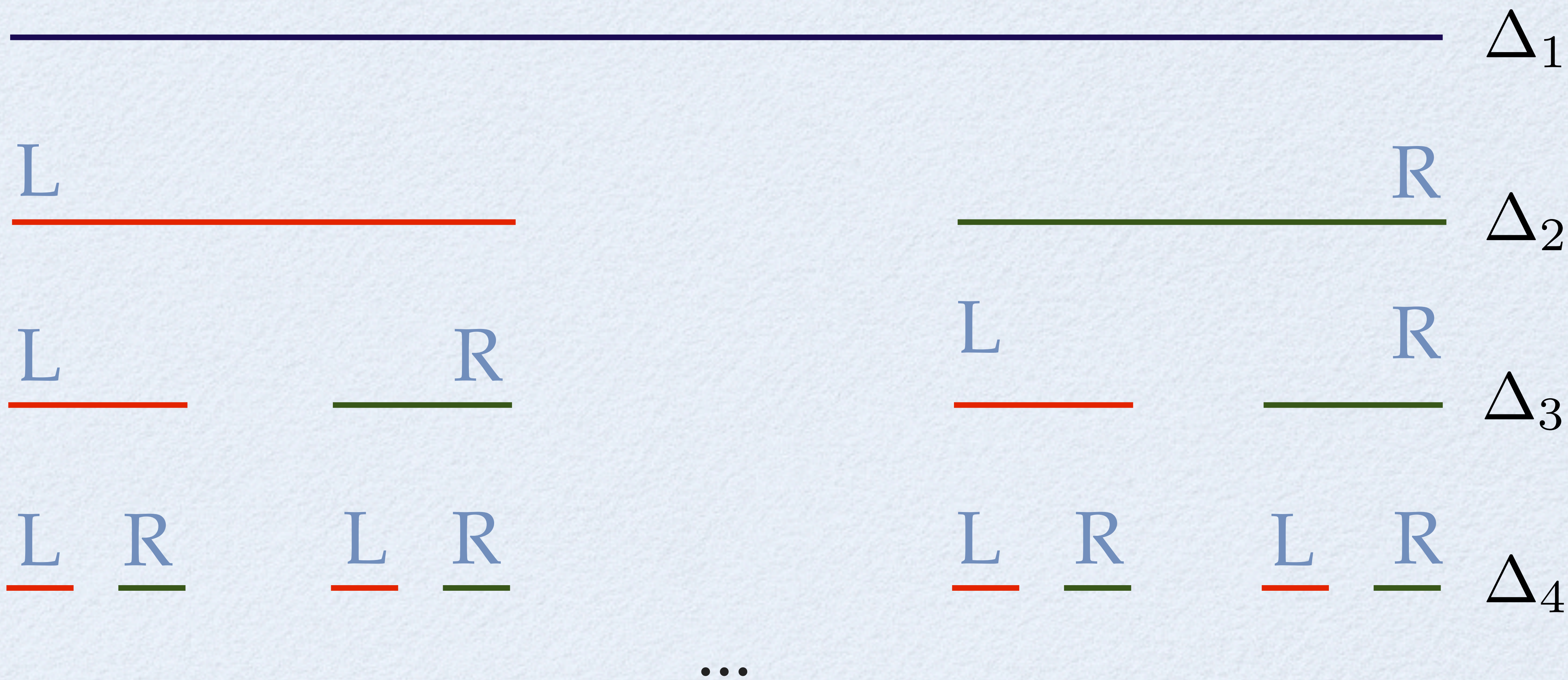

Instead of removing middle thirds, imagine our construction as a process of retaining the left and right thirds of each interval, labeling our choices as we go along.

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we would follow a "path" down this stairstep,
choosing left or right at each step, always careful to
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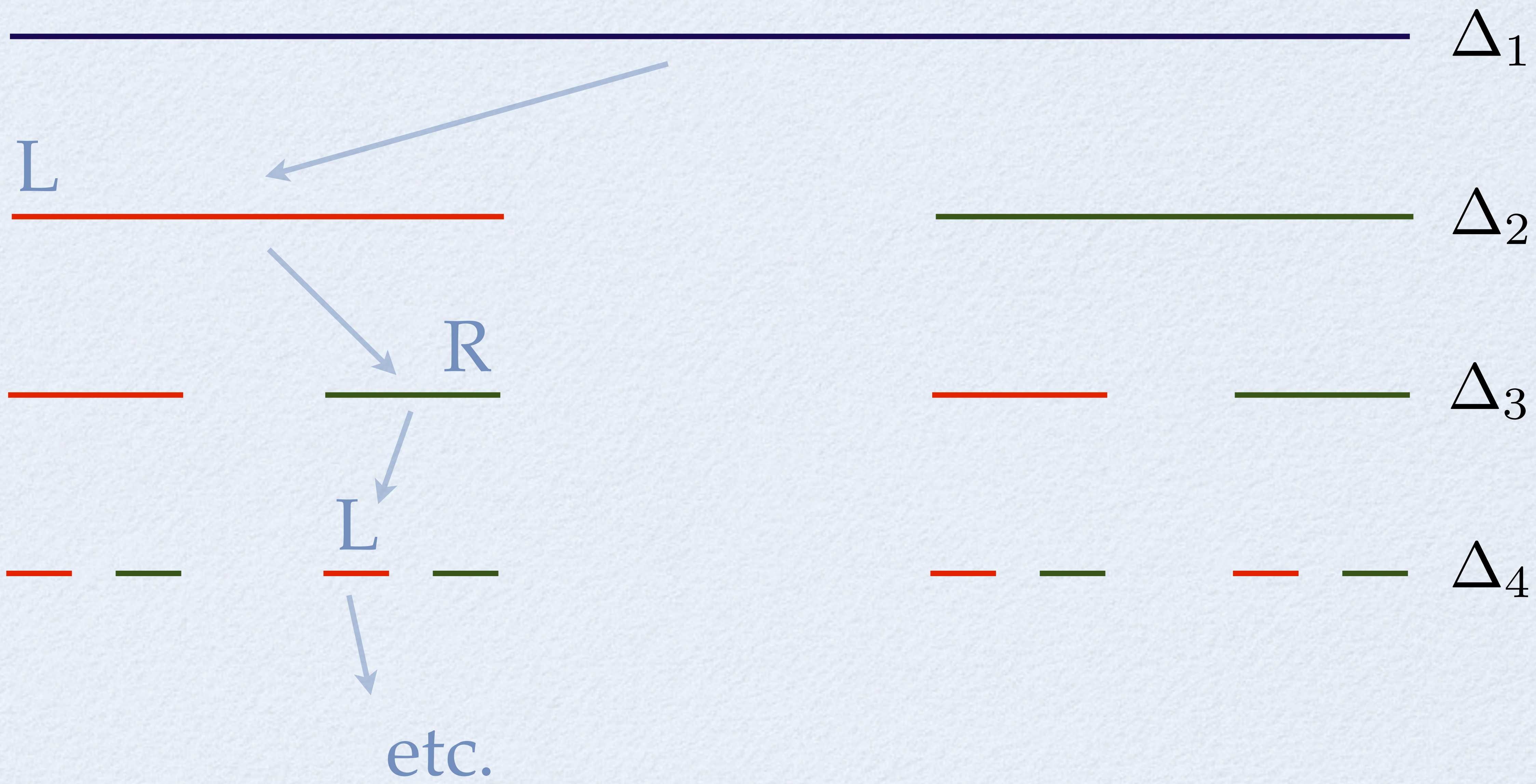
You're not allowed to jump across a gap in the current step;
you just step down, choosing either the left or right third
of the current step.

THE CONSTRUCTION



Any given sequence of choices

LRLLRRRLLLRRR...,

necessarily determines a single point in the Cantor Set, since the corresponding interval "steps" are nested and their lengths decrease to 0.

In other words, the left and right endpoints of our sequence of "steps" will converge to a common value.

Conversely, each point in the Cantor Set uniquely determines the path, or sequence of L's and R's, that leads to it. Indeed, given a point in the Cantor Set, at the "bottom" of our stairstep, there can be but one sequence of interval "steps" that contain it.

But what's so special about L's and R's? Nothing really. Let's use 0's and 1's instead of L's and R's.

But now... a sequence of 0's and 1's

011010001110010...

looks just like a binary decimal

0.011010001110010... (base 2).

Now each point in $[0,1]$ has a binary decimal expansion and, conversely, each binary decimal represents some point in $[0,1]$.

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Thus, we have a correspondence between the points in the Cantor set and the points in the interval [0,1].

We have just shown that the Cantor set is "BIG".

Now we'll show it is "small"!

Let's compute the total length of all the intervals in the Cantor Set.

We will do this by first computing the length of the intervals we removed.

We start with the interval [0,1] and, in the first step, we remove 1 interval of length 1/3; in the second step, we remove 2 more intervals of length 1/9; in the third step we remove 4 more intervals of length 1/27; and so on.

In general, at the n-th stage, we remove another

$$2^{n-1}$$

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Total length of intervals removed:

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots+\frac{2^{n-1}}{3^n}+\cdots=\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$$

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This is a geometric series, and the sum is easy to find.

$$\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}=\frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}}=1$$

Thus, the total length of all the intervals we removed from [0,1] in the construction of the Cantor set equals 1 - the same as the length of the interval we started with!

In other words, we removed everything!?

We know that the Cantor set is a "big" set - after all, it has the same "size" as [0,1] itself, at least in one sense...

...and yet it must also be a "small" set, since it's the result of removing a set of "length 1" from [0,1].

The dilemma, if you will, centers around the fact that we've employed two different notions of "size".

Evidently, "total length" and "cardinality"
are not equivalent notions of size:

a set can be very small in the first sense
while being very large in the second!

based on an article by

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http://personal.bgsu.edu/~carother/cantor/Cantor1.html